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PAPER

Inverse Problems in Science and Engineering Determination of a time-dependent diffusivity in a nonlinear parabolic problem

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In this article, an inverse nonlinear convection-diffusion problem is considered for the identification of an unknown solely time-dependent diffusion coefficient in a subregion of a bounded domain in $\mathrm{R}^d, d \geqslant 1$. The missing data is compensated by boundary observations on a part of the surface of the subdomain: the total flux through that surface or the values of the solution at that surface are measured. Two solution methods are discussed. In both cases, the solvability of the problem is proved using coefficient to data mappings. More specific, a nonlinear numerical algorithm based on Rothe's method is designed and the convergence of approximations towards the weak solution in suitable function spaces is shown. In the proofs, also the monotonicity methods and the Minty-Browder argument are employed. The results of numerical experiments are discussed.

Keywords: Inverse problems, nonlinear parabolic initial boundary value problem; parameter identification; nonlocal boundary condition; time discretization; Minty-Browder; convergence

AMS Subject Classification: 35R30; 35K61; 35B30; 65M32

1. Introduction

Recovery of missing parameters in partial differential equations (PDEs) from overspecified data plays an important role in inverse problems arising in engineering and physics. These problems are widely encountered in the modeling of interesting phenomena, e.g. heat conduction and hydrology. Another challenge of mathematical modelling is to determine what additional information is necessary and/or sufficient to ensure the (unique) solvability of an unknown physical parameter of a given process.

This contribution is devoted to this subjects. More specific, the purpose of this paper is a study on recovery of a time-dependent diffusion coefficient in nonlinear parabolic problems. It is assumed to have some overposed nonlocal data as a side condition. Similar but steady-state settings can be found in the so-called "Spontaneous potential well-logging", which is an important technique to detect parameters of the formation in petroleum exploitation, cf. [1–4]. The determination of a time-dependent diffusivity in parabolic equations has been considered in papers [5, 6]. Nonlinear problems have been studied in [7–10]. The problems studied in these papers are all one-dimensional in space. However, the analysis made in this article is valid for every dimension $d \geqslant 1$.

The mathematical setting is the following. Let $\Omega \subset \mathrm{R}^d$ be a bounded domain with a Lipschitz continuous boundary Γ. The domain Ω is split into two nonoverlapping parts $\Omega_0 \neq \emptyset$ and $\Omega \setminus \overline{\Omega}_0$ with the assumption that $\Omega \setminus \overline{\Omega}_0$ cannot surround Ω_0. A transient

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convection-diffusion process is considered in Ω. The diffusion coefficient K takes the form $K = k(t, \boldsymbol{x})\kappa(t, \boldsymbol{x})$ for a known κ and $k(t, \boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \Omega \setminus \overline{\Omega}_0$ and $k(t, \boldsymbol{x}) = k(t)$ for $\boldsymbol{x} \in \Omega_0$. The boundary Γ is split into three nonoverlapping parts, namely Γ_N (Neumann part), Γ_D (Dirichlet part) and $\Gamma_0 \subset \Gamma \cap \partial\Omega_0$, on which beside a Dirichlet boundary condition (BC) also the total flux through this part is prescribed. It is assumed that $\overline{\Gamma}_D \cap \overline{\Gamma}_0 = \emptyset$ with $\mu(\Gamma_0) > 0$, where μ is the Lebesgue measure. The purpose of this article is to study the following nonlinear parabolic initial boundary value problem (IBVP) (1)-(2): find a couple (K, u) such that ($T > 0$ fixed)

$$\begin{cases} \partial_t\theta(u) - \nabla \cdot (K\nabla u + \boldsymbol{a}(u)) = f & \text{in } Q_T := (0,T) \times \Omega; \\ u = g^D & \text{in } (0,T) \times \Gamma_D; \\ (-K\nabla u - \boldsymbol{a}(u)) \cdot \boldsymbol{\nu} = g^N & \text{in } (0,T) \times \Gamma_N; \\ u(0) = u_0 & \text{in } \Omega; \end{cases} \tag{1}$$

and such that the following boundary measurements are satisfied

$$\begin{cases} \int_{\Gamma_0} (-K\nabla u - \boldsymbol{a}(u)) \cdot \boldsymbol{\nu} = h(t) \ \text{ in } (0,T); \\ u = U(t) \text{ on } (0,T) \times \Gamma_0. \end{cases} \tag{2}$$

More specific, the inverse problem of determining the unknown coefficient K from the measured data (2) is considered. It is determined under which assumpions on the data this inverse problem has a weak solution (K, u). Also a nonlinear numerical algorithm based on backward Euler method is designed to approximate the solution (K, u) and the convergence of approximations towards the weak solution in suitable function spaces is shown. An easier case with $\theta(u) = u$ and $\boldsymbol{a} \equiv \boldsymbol{0}$ is studied in [11].

The remainder of this paper is organized as follows. Section 2 summarizes the mathematical tools and the assumptions on the data that are needed. Two different solution methods are presented into detail in Section 3. The existence of a solution to (1)-(2) for each solution method is shown in Section 4. Finally, some numerical experiments are developed in Section 5.

2. Variational setting and assumptions

First, some standard notations are introduced. The euclidian norm of a vector $\boldsymbol{v}$ in R^d is denoted by $|\boldsymbol{v}|$. To increase the readability of the text, it is assumed without loss of generality that $g^D = 0$ and $g^N = 0$. The suitable choice of a test space is

$$V := \{\varphi \in H^1(\Omega);\ \varphi|_{\Gamma_D} = 0,\ \varphi|_{\Gamma_0} = \mathit{constant}\},$$

which is clearly a Hilbert space with norm $\|u\|_V^2 = \|u\|^2 + \|\nabla u\|^2$, where $\|\cdot\|$ represents the norm in $L_2(\Omega)$. Also a subspace of V is considered, namely

$$W := \{\varphi \in H^1(\Omega);\ \varphi|_{\Gamma_0 \cup \Gamma_D} = 0\}.$$

Due to the homogeneous Dirichlet boundary condition (BC) on the boundary Γ_D, the following Friedrichs inequality holds for every function $u \in V$

$$\|u\|_V \leqslant C \|\nabla u\|. \tag{3}$$

Consequently, the norm in the space V is equivalent with the seminorm $\|\nabla u\|$. The norm in the trace space $L_2(\gamma)$, with $\gamma \subset \Gamma$, is written by $\|\cdot\|_{L_2(\gamma)}$. The dual space of V is denoted by V^*.

What now follows is a list of assumptions that are necessary to prove the existence of a weak solution to the nonlinear problem (1)-(2) on a single time step. The continuous function $\theta : \mathrm{R} \to \mathrm{R}$ (diffusion term) has to satisfy

$$\theta(0) = 0; \tag{4}$$

$$|\theta(s)| \leqslant C(1 + |s|) \qquad \text{a.e. in R.} \tag{5}$$

Moreover,

$$\begin{cases} \theta'(s) \geqslant 0 & \text{a.e. in R if } \boldsymbol{a}(u) = \boldsymbol{a} \ (\boldsymbol{a} \text{ independent of } u); \\ \theta'(s) \geqslant \theta_0 > 0 & \text{a.e. in R if } \boldsymbol{a}(u) \neq \boldsymbol{a} \ (\boldsymbol{a} \text{ depends on } u). \end{cases} \tag{6}$$

Remark 2.1. The condition $\boldsymbol{a}(u(t, \mathbf{x})) = \boldsymbol{a}(t, \mathbf{x})$ in (6) means that $\boldsymbol{a}$ does not depend on the solution u, but only on time and position, i.e. $(t, \mathbf{x}) \in Q_T$. If $\boldsymbol{a}(u) \neq \boldsymbol{a}$, then $\boldsymbol{a}$ is depending on u, for instance $\boldsymbol{a}(u) = \boldsymbol{b}u$ with $\boldsymbol{b} : Q_T \to \mathrm{R}^d$.

The vector-valued function $\boldsymbol{a} : \mathrm{R} \to \mathrm{R}^d$ (convection term) is also continuous and

$$|\boldsymbol{a}(s)| = |\,(\boldsymbol{a}_1(s), \boldsymbol{a}_2(s), \dots, \boldsymbol{a}_d(s))\,| \leqslant C \qquad \text{a.e. in R;} \tag{7}$$

$$|\boldsymbol{a}'(s)| = |\,(\boldsymbol{a}_1'(s), \boldsymbol{a}_2'(s), \dots, \boldsymbol{a}_d'(s))\,| \leqslant C \qquad \text{a.e. in R.} \tag{8}$$

Remark 2.2. If $\boldsymbol{a}(u) = \boldsymbol{a}$, then the right-hand side (RHS) f of (1) can be redefined as $f + \nabla \cdot \boldsymbol{a}$. From now on, it is assumed that $\nabla \cdot \partial_t \boldsymbol{a} \in L_2((0, T), L_2(\Omega))$ such that the $\boldsymbol{a}(u)$-term in problem (1)-(2) can be cancelled out of the equations if $\boldsymbol{a}(u) = \boldsymbol{a}$, see also equation (14). Therefore, assumptions (7)-(8) are only necessary if $\boldsymbol{a}(u) \neq \boldsymbol{a}$.

Remark 2.3. The source function f in problem (1)-(2) depends only on the time and space variable. However, our numerical procedure can be generalized to a nonlinear $f(u)$, assuming $\theta' \geqslant \theta_0 > 0$ if $\boldsymbol{a}(u) = \boldsymbol{a}$. The purpose of this paper is to keep the regularity as low as possible during the analysis. This is the reason why is started with the lowest plausible assumptions. Further, during the analysis, more assumptions are necessary on the function θ.

The following assumptions on the other data functions are adopted

$$0 < C_0 \leqslant k \leqslant C_1; \tag{9}$$

$$0 < D_0 \leqslant \kappa \leqslant D_1; \tag{10}$$

$$\kappa \in C([0, T], H^1(\Omega)); \tag{11}$$

$$U \in C([0, T]); \tag{12}$$

$$h' \in L_2((0, T)) \Rightarrow h \in C([0, T]); \tag{13}$$

$$\partial_t f \in L_2((0, T), L_2(\Omega)) \Rightarrow f \in C([0, T], L^2(\Omega)). \tag{14}$$

The initial datum satisfies

$$u_0 \in H^1(\Omega). \tag{15}$$

Note that the data function U can be prolonged into the whole domain Ω by a function $\tilde{U}$ in such a way that [12, Lemma 5.1]

$$\tilde{U} \in C\left([0,T], H^1(\Omega)\right), \qquad \tilde{U} = \begin{cases} 0 & \text{on } [0,T] \times \Gamma_D; \\ U & \text{on } [0,T] \times \Gamma_0. \end{cases} \tag{16}$$

Finally, some useful (in)equalities are stated, which can easily be derived:

$$2\sum_{i=1}^{n}(a_i - a_{i-1})a_i = a_n^2 - a_0^2 + \sum_{i=1}^{n}(a_i - a_{i-1})^2, \qquad a_i \in \mathrm{R}; \qquad \text{Abel's summation rule}$$

and

$$ab \leqslant \varepsilon a^2 + C_\varepsilon b^2, \qquad a, b \in \mathrm{R}, \quad \varepsilon > 0. \qquad \text{Young's inequality}$$

Remark 2.4. In this contribution, the values $C, \varepsilon, C_\varepsilon$ are generic and positive constants independent of the discretization parameter τ, see Section 3. They can be different from place to place. The value ε is small and $C_\varepsilon = C\left(\varepsilon^{-1}\right)$. To reduce the number of arbitrary constants, we use the notation $a \lesssim b$ if there exists a positive constant C such that $a \leqslant Cb$.

3. A single time step

Rothe's method [13] is applied to prove the existence of a weak solution to problem (1)-(2). An equidistant time-partitioning is used with timestep $\tau = T/n < 1$, for any $n \in \mathrm{N}$. Let us introduce the notations $t_i = i\tau$ and for any function z

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

The following recursive approximation scheme is proposed for $i = 1, \ldots, n$: find the unknown couple $(k_i, u_i) \in \mathrm{R}_+ \times V$ on each time step that satisfies

$$\begin{cases} \delta\theta(u_i) - \nabla \cdot (K_i \nabla u_i + \boldsymbol{a}(u_i)) = f_i & \text{in } \Omega; \\ u_i = 0 & \text{on } \Gamma_D; \\ (-K_i \nabla u_i - \boldsymbol{a}(u_i)) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_N; \\ \int_{\Gamma_0} (-K_i \nabla u_i - \boldsymbol{a}(u_i)) \cdot \boldsymbol{\nu} = h_i & \\ u_i = U_i & \text{on } \Gamma_0, \end{cases} \tag{17}$$

where $K_i = k_i \kappa_i$. In this section, the existence of (K_i, u_i) for any $i = 1, \ldots, n$ is shown in two ways: two different methods for solving (17) are presented.

In the first one, it is assumed that k_i is given and there is looked for a solution of the direct problem

$$
\begin{cases}
\delta\theta(u_i) - \nabla \cdot (K_i \nabla u_i + \boldsymbol{a}(u_i)) = f_i & \text{in } \Omega; \\
u_i = 0 & \text{on } \Gamma_D; \\
(-K_i \nabla u_i - \boldsymbol{a}(u_i)) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_N; \\
\int_{\Gamma_0} (-K_i \nabla u_i - \boldsymbol{a}(u_i)) \cdot \boldsymbol{\nu} = h_i. &
\end{cases}
\tag{18}
$$

Afterwards, it is shown that the trace of u_i on Γ_0 depends continuously on k_i. Then, k_i is determined such that $u_i|_{\Gamma_0} = U_i$. This solution method is discussed into detail in Subsection 3.1.

In the second solution method, the following direct problem is solved

$$
\begin{cases}
\delta\theta(u_i) - \nabla \cdot (K_i \nabla u_i + \boldsymbol{a}(u_i)) = f_i & \text{in } \Omega; \\
u_i = 0 & \text{on } \Gamma_D; \\
(-K_i \nabla u_i - \boldsymbol{a}(u_i)) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_N; \\
u_i = U_i & \text{on } \Gamma_0
\end{cases}
\tag{19}
$$

for a given k_i. It is proved that the total flux through Γ_0, $\int_{\Gamma_0} (-K_i \nabla u_i - \boldsymbol{a}(u_i)) \cdot \boldsymbol{\nu}$, depends continuously on k_i. At the end, k_i is found such that $\int_{\Gamma_0} (-K_i \nabla u_i - \boldsymbol{a}(u_i)) \cdot \boldsymbol{\nu} = h_i$. The reader can find more details in Subsection 3.2. Only the differences with the previous solution method are pointed out.

Finally, in Subsection 3.3 a lemma is stated about the existence of a solution on a single time step of problem (17). This lemma is based on the results of Subsections 3.1 and 3.2.

3.1 First solution method

First, the variational formulation of (18) is considered

$$
\frac{1}{\tau} (\theta(u_i), \varphi) + (K_i \nabla u_i + \boldsymbol{a}(u_i), \nabla \varphi) = (f_i, \varphi) - h_i \varphi|_{\Gamma_0} + \frac{1}{\tau} (\theta(u_{i-1}), \varphi), \quad \varphi \in V. \tag{20}
$$

In the following lemma, the theory of monotone operators is applied to prove the existence of a weak solution to (20) for given K_i. The interested reader is referred to [14] for further information.

Lemma 3.1 (Unicity). *Assume (4)-(15). For any given $k_i > 0$, $i = 1, \ldots, n$, there exist a $\tau_0 > 0$ and an uniquely determined $u_{k_i} \in V$ solving (18) for $\tau < \tau_0$.*

Proof. We consider the nonlinear operators $A_i : V \to V^*, i = 1, \ldots, n$, defined as

$$
\langle A_i(u), \varphi \rangle := \frac{1}{\tau} (\theta(u), \varphi) + (K_i \nabla u + \boldsymbol{a}(u), \nabla \varphi),
$$

together with the linear functionals $F_i : V \to \mathrm{R}, i = 1, \ldots, n$, such that

$$
\langle F_i, \varphi \rangle := (f_i, \varphi) - h_i \varphi|_{\Gamma_0} + \frac{1}{\tau} (\theta(u_{i-1}), \varphi).
$$

We proof that A_i is strictly monotone, coercive and demicontinuous. In the case of $\boldsymbol{a}(u) =$

$\boldsymbol{a}$, it follows from the monotonicity of θ that

$$\begin{aligned}\langle A_i(u) - A_i(v), u - v\rangle &= \frac{1}{\tau}\left(\theta(u) - \theta(v), u - v\right) + \left(K_i\nabla(u - v), \nabla(u - v)\right)\\ &\geqslant C_0 D_0 \left\|\nabla(u - v)\right\|^2.\end{aligned}$$

The strict monotonicity of A_i follows by an application of the Friedrichs inequality (3). Secondly, we do the case that $\boldsymbol{a}(u) \neq \boldsymbol{a}$. Using the mean value theorem and the Young's inequality, we have

$$\begin{aligned}&\langle A_i(u) - A_i(v), u - v\rangle\\ &= \frac{1}{\tau}\left(\theta'(\xi_1)[u - v], u - v\right) + \left(K_i\nabla(u - v), \nabla(u - v)\right) + \left(\boldsymbol{a}'(\xi_2)[u - v], \nabla(u - v)\right)\\ &\geqslant \frac{\theta_0}{\tau}\left\|u - v\right\|^2 + C_0 D_0 \left\|\nabla(u - v)\right\|^2 - C_\varepsilon \left\|\boldsymbol{a}'(\xi_2)[u - v]\right\|^2 - \varepsilon\left\|\nabla(u - v)\right\|^2\\ &\geqslant \left(\frac{\theta_0}{\tau} - C_\varepsilon\right)\left\|u - v\right\|^2 + \left(C_0 D_0 - \varepsilon\right)\left\|\nabla(u - v)\right\|^2.\end{aligned}$$

Fixing a sufficiently small $\varepsilon > 0$ and $\tau(\varepsilon) > 0$ gives the strict monotonicity of the operator A_i. The operator A_i is coercive if

$$\lim_{\|u\|_V \to +\infty} \frac{\langle A_i(u), u\rangle}{\|u\|_V} = +\infty.$$

This is always satisfied. If $\boldsymbol{a}(u) \neq \boldsymbol{a}$, then the following lower bound is valid for sufficiently small ε

$$\begin{aligned}\langle A_i(u), u\rangle &= \frac{1}{\tau}\left(\theta(u), u\right) + \left(K_i\nabla u + \boldsymbol{a}(u), \nabla u\right)\\ &\overset{(4)}{\geqslant} \frac{\theta_0}{\tau}\left\|u\right\|^2 + C_0 D_0 \left\|\nabla u\right\|^2 - C_\varepsilon\left\|\boldsymbol{a}(u)\right\|^2 - \varepsilon\left\|\nabla u\right\|^2\\ &\overset{(7)}{\geqslant} C\left\|u\right\|_V^2 - C.\end{aligned}$$

In the case $\boldsymbol{a}(u) = \boldsymbol{a}$, the second constant in this lower bound disappears

$$\langle A_i(u), u\rangle \geqslant C_0 D_0 \left\|\nabla u\right\|^2 \overset{(3)}{\geqslant} C\left\|u\right\|_V^2.$$

The demicontinuity of A_i follows from the continuity of θ and $\boldsymbol{a}$. The functionals F_i belong to V^* if $u_{i-1} \in L_2(\Omega)$ due to $f_i \in L_2(\Omega)$, the boundedness of h, the trace theorem (cf. [15, Theorem 3.9]) and the growth condition on θ

$$\left|\langle F_i, \varphi\rangle\right| \leqslant \left\|f_i\right\|\left\|\varphi\right\| + |h_i|\frac{\|\varphi\|_{L_2(\Gamma_0)}}{\sqrt{|\Gamma_0|}} + \frac{1}{\tau}\left\|\theta(u_{i-1})\right\|\left\|\varphi\right\| \lesssim \left(1 + \left\|u_{i-1}\right\|\right)\left\|\varphi\right\|_V.$$

Consequently, since $u_0 \in L_2(\Omega)$, the equation $A_i(u) = F_i$ admits a unique solution for $i = 1, \ldots, n$. $\square$

In the subsequent Lemma, the existence of an uniform bound for u_{k_i} independent of k_i is proved.

Lemma 3.2 (Uniform bound for u_{k_i}). *Assume (4)-(15). There exists a positive constant C such that*

$$\|u_{k_i}\|_V^2 \leqslant \frac{C}{\tau^2} \qquad \text{for } C_0 \leqslant k_i \leqslant C_1, \quad i = 1, \ldots, n.$$

Proof. We consider $k_i > 0$ as a parameter. We set $\varphi = u_{k_i}$ into (20) and get

$$\begin{aligned}
&\frac{1}{\tau}\left(\theta(u_{k_i}), u_{k_i}\right) + \left(K_i \nabla u_{k_i}, \nabla u_{k_i}\right) \\
&= \left(f_i, u_{k_i}\right) - h_i u_{k_i}|_{\Gamma_0} + \frac{1}{\tau}\left(\theta(u_{i-1}), u_{k_i}\right) - \left(\boldsymbol{a}(u_{k_i}), \nabla u_{k_i}\right).
\end{aligned}$$

Remark that the solution on the previous timestep, u_{i-1}, belongs to $L_2(\Omega)$. First, we consider the case $\boldsymbol{a}(u) = \boldsymbol{a}$. Using the Cauchy's and Young's inequalities together with the growth condition on θ, the trace theorem and the Friedrichs inequality (3) we obtain

$$\begin{aligned}
\left|\left(f_i, u_{k_i}\right)\right| &\leqslant C_\varepsilon \|f_i\|^2 + \varepsilon \|u_{k_i}\|^2 \leqslant C_\varepsilon + \varepsilon \|\nabla u_{k_i}\|^2, \\
\left|h_i u_{k_i}|_{\Gamma_0}\right| &\leqslant C_\varepsilon h_i^2 + \varepsilon \frac{\|u_{k_i}\|^2_{L_2(\Gamma_0)}}{|\Gamma_0|} \leqslant C_\varepsilon + \varepsilon \|\nabla u_{k_i}\|^2, \\
\left|\frac{1}{\tau}\left(\theta(u_{i-1}), u_{k_i}\right)\right| &\leqslant \frac{C_\varepsilon}{\tau^2} \|\theta(u_{i-1})\|^2 + \varepsilon \|u_{k_i}\|^2 \leqslant \frac{C_\varepsilon}{\tau^2} + \varepsilon \|\nabla u_{k_i}\|^2.
\end{aligned}$$

Using the monotonicity of θ and the uniform bounds (9) and (10) one can easily get

$$\left(C_0 D_0 - \varepsilon\right) \|\nabla u_{k_i}\|^2 \leqslant \frac{C_\varepsilon}{\tau^2}.$$

Analogue, in the case $\boldsymbol{a}(u) \neq \boldsymbol{a}$, we can readily obtain

$$\left|\left(\boldsymbol{a}(u_{k_i}), \nabla u_{k_i}\right)\right| \leqslant C_\varepsilon \|\boldsymbol{a}(u_{k_i})\|^2 + \varepsilon \|\nabla u_{k_i}\|^2 \overset{(7)}{\leqslant} C_\varepsilon + \varepsilon \|\nabla u_{k_i}\|^2$$

and therefore

$$\frac{\theta_0}{\tau} \|u_{k_i}\|^2 + \left(C_0 D_0 - \varepsilon\right) \|\nabla u_{k_i}\|^2 \leqslant \frac{C_\varepsilon}{\tau^2}.$$

Fixing a sufficiently small positive ε in both cases and applying the Friedrichs inequality if $\boldsymbol{a}(u) = \boldsymbol{a}$, we conclude the proof. ☐

To prove that the trace of u_i on Γ_0 depends continuously on k_i, it is necessary to define the following coefficient to data map

$$\mathcal{T} : [C_0, C_1] \to \mathrm{R} : k_i \mapsto \mathcal{T}(k_i) := u_{k_i}|_{\Gamma_0}.$$

Then, the inverse problem with the measured output data $U(t)$ can be formulated as the following operator equation: search k_i such that

$$\mathcal{T}(k_i) = U_i.$$

The continuity of this input-output map $\mathcal{T}$ is investigated in the following lemma. It is this property that leads to the existence of a solution to problem (17), see Lemma 3.7.

Lemma 3.3 (u_{k_i} depends continuously on k_i)**.** *Assume (4)-(15). There exists a $\tau_0 > 0$ such that the function $\mathcal{T}$ is continuous for $\tau < \tau_0$.*

Proof. Subtract (20) for $k_i = \beta$ from (20) for $k_i = \alpha$ and set $\varphi = u_\alpha - u_\beta$ to get

$$\begin{aligned}&\frac{1}{\tau}\left(\theta(u_\alpha) - \theta(u_\beta), u_\alpha - u_\beta\right) + \left(\alpha\kappa_i\nabla(u_\alpha - u_\beta), \nabla(u_\alpha - u_\beta)\right)\\ &= \left((\beta-\alpha)\kappa_i\nabla u_\beta, \nabla(u_\alpha - u_\beta)\right) - \left(\boldsymbol{a}(u_\alpha) - \boldsymbol{a}(u_\beta), \nabla(u_\alpha - u_\beta)\right).\end{aligned}$$

The first term in the RHS can be bounded using the Cauchy's and Young's inequality, Lemma 3.2 and the uniform bound (10)

$$\begin{aligned}\left((\beta-\alpha)\kappa_i\nabla u_\beta, \nabla(u_\alpha - u_\beta)\right) &\leqslant C_\varepsilon \left\|(\beta-\alpha)\kappa_i\nabla u_\beta\right\|^2 + \varepsilon\left\|\nabla(u_\alpha - u_\beta)\right\|^2\\ &\leqslant \frac{C_\varepsilon}{\tau^2}(\beta-\alpha)^2 + \varepsilon\left\|\nabla(u_\alpha - u_\beta)\right\|^2.\end{aligned}$$

The second term in the RHS disappears if $\boldsymbol{a}(u) = \boldsymbol{a}$. An obvious calculation employing the monotonicity of θ gives

$$(\alpha D_0 - \varepsilon)\left\|\nabla(u_\alpha - u_\beta)\right\|^2 \leqslant \frac{C_\varepsilon}{\tau^2}(\alpha-\beta)^2.$$

We use the mean value theorem in the case of $\boldsymbol{a}(u) \neq \boldsymbol{a}$ to get

$$\begin{aligned}\left(\boldsymbol{a}(u_\alpha) - \boldsymbol{a}(u_\beta), \nabla(u_\alpha - u_\beta)\right) &\leqslant C_\varepsilon\left\|\boldsymbol{a}(u_\alpha) - \boldsymbol{a}(u_\beta)\right\|^2 + \varepsilon\left\|\nabla(u_\alpha - u_\beta)\right\|^2\\ &\leqslant C_\varepsilon\left\|u_\alpha - u_\beta\right\|^2 + \varepsilon\left\|\nabla(u_\alpha - u_\beta)\right\|^2.\end{aligned}$$

Therefore, we can derive the following estimate

$$\left(\frac{\theta_0}{\tau} - C_\varepsilon\right)\left\|u_\alpha - u_\beta\right\|^2 + (\alpha D_0 - \varepsilon)\left\|\nabla(u_\alpha - u_\beta)\right\|^2 \leqslant \frac{C_\varepsilon}{\tau^2}(\alpha-\beta)^2.$$

We fix a sufficiently small ε and $\tau(\varepsilon)$ to conclude that in both cases

$$\left\|\nabla(u_\alpha - u_\beta)\right\|^2 \lesssim \frac{(\alpha-\beta)^2}{\tau^2}. \tag{21}$$

Using the trace theorem and the Friedrichs inequality, we deduce in both cases that

$$|\mathcal{T}(\alpha) - \mathcal{T}(\beta)| = \frac{\left\|u_\alpha - u_\beta\right\|_{L_2(\Gamma_0)}}{\sqrt{|\Gamma_0|}} \lesssim \left\|\nabla(u_\alpha - u_\beta)\right\| \overset{(21)}{\lesssim} \frac{1}{\tau}|\alpha - \beta|.$$

$\square$

3.2 Second solution method

In contrast to the first solution method, the Lipschitz continuity of θ is needed in the second solution method, i.e. there exists a real constant θ_1 such that

$$\theta'(s) \leqslant \theta_1 \qquad \text{a.e. in R.} \tag{22}$$

To get rid of the nonhomogeneous Dirichlet boundary condition on Γ_0, the solution of (19) is prescribed as $u_i := v_i + \tilde{U}_i$, $i = 1, \ldots, n$, where v_i is unknown. Next, define the functions $\vartheta : \mathrm{R} \to \mathrm{R}$ and $\boldsymbol{b} : \mathrm{R} \to \mathrm{R}^d$ by setting

$$\vartheta(s) = \theta(s + \tilde{U}_i) \qquad \text{and} \qquad \boldsymbol{b}(s) = \boldsymbol{a}(s + \tilde{U}_i).$$

Thanks to the properties of θ and $\boldsymbol{a}$, the function ϑ is monotonically increasing if $\boldsymbol{b}$ is independent of v_i and strict monotonically increasing if $\boldsymbol{b}$ depends on v_i. Using the preceding assumptions, the variational formulation of (19) can be rewritten (for given K_i) for all $\varphi \in W$ as

$$\frac{1}{\tau}\left(\vartheta(v_i), \varphi\right) + \left(K_i \nabla v_i + \boldsymbol{b}(v_i), \nabla\varphi\right) = (f_i, \varphi) - \left(K_i \nabla \tilde{U}_i, \nabla\varphi\right) + \frac{1}{\tau}\left(\vartheta(v_{i-1}), \varphi\right). \quad (23)$$

The following growth condition on ϑ is satisfied due to the growth condition on θ

$$\|\vartheta(v)\| \lesssim 1 + \left\|v + \tilde{U}_i\right\| \overset{(16)}{\lesssim} 1 + \|v\|, \qquad v : \Omega \to \mathrm{R}.$$

Analogue as in the previous subsection one can state three lemmas.

Lemma 3.4 (Unicity). *Assume (4)-(16). For any given $k_i > 0$, $i = 1, \ldots, n$, there exist a $\tau_0 > 0$ and an uniquely determined $v_{k_i} \in W$ solving (23) for $\tau < \tau_0$. Moreover, the function $u_{k_i} = v_{k_i} + \tilde{U}_i \in V$ is solving (19).*

Proof. This is almost an exact analogue of the proof of Lemma 3.1, except for the appearance of the following lower bound that is valid for each $v \in W$

$$\begin{aligned}
(\vartheta(v), v) &= \left(\theta(v + \tilde{U}_i), v + \tilde{U}_i\right) - \left(\theta(v + \tilde{U}_i), \tilde{U}_i\right) \\
&\geqslant -\left|\left(\theta(v + \tilde{U}_i), \tilde{U}_i\right)\right| \\
&\geqslant -C\left(1 + \|v\| + \left\|\tilde{U}_i\right\|\right)\left\|\tilde{U}_i\right\| \\
&\overset{(3)}{\geqslant} -C_\varepsilon - \varepsilon \|\nabla v\|^2. \qquad (24)
\end{aligned}$$

□

Lemma 3.5 (Uniform bound for u_{k_i}). *Assume (4)-(16). There exists a positive constant C such that for $i = 1, \ldots, n$*

$$\|v_{k_i}\|_V^2 \leqslant \frac{C}{\tau^2} \qquad \textit{for } C_0 \leqslant k_i \leqslant C_1, \quad i = 1, \ldots, n.$$

Proof. The proof follows very closely the proof of Lemma 3.2 using the lower bound (24). □

At this point, define the coefficient to data map Ψ by

$$\Psi : [C_0, C_1] \to \mathrm{R} : k_i \mapsto \Psi(k_i) := -\int_{\Gamma_0} \left(k_i \kappa_i \nabla u_{k_i} + \boldsymbol{a}(u_{k_i})\right) \cdot \boldsymbol{\nu}.$$

Now, the inverse problem with the measured output data $h(t)$ can be formulated as the following operator equation: search k_i such that

$$\Psi(k_i) = h_i.$$

In the following lemma, the continuity of the map $\mathcal{T}$ is proved.

Lemma 3.6 (u_{k_i} depends continuously on k_i)**.** *Assume (4)-(16). There exists a $\tau_0 > 0$ such that the function Ψ is continuous for $\tau < \tau_0$.*

Proof. Subtract (23) for $k_i = \beta$ from (23) for $k_i = \alpha$ and set $\varphi = v_\alpha - v_\beta$ to get

$$\begin{aligned}
&\frac{1}{\tau}\left(\vartheta(v_\alpha) - \vartheta(v_\beta), v_\alpha - v_\beta\right) + \left(\alpha\kappa_i \nabla(v_\alpha - v_\beta), \nabla(v_\alpha - v_\beta)\right) \\
&= \left((\beta-\alpha)\kappa_i \nabla v_\beta, \nabla(v_\alpha - v_\beta)\right) + \Big((\beta-\alpha)\kappa_i \nabla \tilde{U}_i, \nabla(v_\alpha - v_\beta)\Big) \\
&\quad - \left(\boldsymbol{b}(v_\alpha) - \boldsymbol{b}(v_\beta), \nabla(v_\alpha - v_\beta)\right),
\end{aligned}$$

which implies using Lemma 3.5

$$\|\nabla(v_\alpha - v_\beta)\|^2 \lesssim \frac{(\alpha-\beta)^2}{\tau^2}. \tag{25}$$

Recall that $\overline{\Gamma}_D \cap \overline{\Gamma}_0 = \emptyset$. Hence, the existence of a function $\Phi \in C^\infty(\overline{\Omega})$ such that $\Phi|_{\Gamma_D} = 0$ and $\Phi|_{\Gamma_0} = 1$ is guaranteed by Friedman [12, Lemma 5.1]. Using problem (19), we have that

$$\begin{aligned}
\Psi(k_i) &= -\left((k_i\kappa_i \nabla u_{k_i} + \boldsymbol{a}(u_{k_i})) \cdot \boldsymbol{\nu}, 1\right)_{\Gamma_0} \\
&= (f_i, \Phi) - \frac{1}{\tau}\left(\theta(u_{k_i}), \Phi\right) - \left(k_i\kappa_i \nabla u_{k_i} + \boldsymbol{a}(u_{k_i}), \nabla\Phi\right) + \frac{1}{\tau}\left(\theta(u_{i-1}), \Phi\right).
\end{aligned}$$

Therefore, using the mean value theorem we deduce that

$$\begin{aligned}
&|\Psi(\alpha) - \Psi(\beta)| \\
&= |\tau^{-1}\left(\theta(u_\alpha) - \theta(u_\beta), \Phi\right) + \left(\boldsymbol{a}(u_\alpha) - \boldsymbol{a}(u_\beta), \nabla\Phi\right) \\
&\qquad + \left(\alpha\kappa_i \nabla(u_\alpha - u_\beta), \nabla\Phi\right) + \left((\alpha-\beta)\kappa_i \nabla u_\beta, \nabla\Phi\right)| \\
&\overset{(22)}{\lesssim} \frac{\theta_1}{\tau}\|v_\alpha - v_\beta\|\,\|\Phi\| + \|v_\alpha - v_\beta\|\,\|\nabla\Phi\| \\
&\qquad + \|\nabla(v_\alpha - v_\beta)\|\,\|\nabla\Phi\| + |\alpha-\beta|\left(\|\nabla v_\beta\| + \left\|\nabla\tilde{U}_i\right\|\right)\|\nabla\Phi\| \\
&\overset{(3)}{\lesssim} \frac{1}{\tau}\left(\|\nabla(v_\alpha - v_\beta)\| + |\alpha-\beta|\right) \\
&\overset{(25)}{\lesssim} \frac{|\alpha-\beta|}{\tau^2}.
\end{aligned}$$

This completes the proof. □

3.3 Solvability of (17)

The following Lemma is a consequence of subsections 3.1 and 3.2.

Lemma 3.7. *Assume (4)-(16). If $U(t) \in \mathcal{T}([C_0, C_1])$ $\forall t \in [0, T]$, then there exist a $\tau_0 > 0$ and a couple $(k_i, u_i) \in \mathrm{R}_+ \times V$ that solves (17) for $\tau < \tau_0$. If θ is Lipschitz continuous and $h(t) \in \Psi([C_0, C_1])$ $\forall t \in [0, T]$, then there exist a $\tau_0 > 0$ and a couple $(k_i, u_i) \in \mathrm{R}_+ \times V$ that solves (17) for $\tau < \tau_0$.*

4. Convergence

In this section, the stability estimates are derived. Afterwards, there is passed to the limit for $n \to \infty$ to get the existence of a solution to (1)-(2). Again, the two different solution methods are considered.

4.1 *First solution method*

The variational formulation of (18) on timestep t_i reads as

$$\begin{aligned} (\delta\theta(u_i), \varphi) + (K_i \nabla u_i + \boldsymbol{a}(u_i), \nabla\varphi) + h_i \varphi|_{\Gamma_0} &= (f_i, \varphi) \qquad \varphi \in V \\ u_i|_{\Gamma_0} &= U_i. \end{aligned} \tag{26}$$

The formulation (26) has a solution on t_i according to Lemma 3.7. The next step is the stability analysis. First, two functions are introduced, which simplifies the proofs. Let γ be any monotone increasing real function with $\gamma(0) = 0$. A primitive function of γ is denoted by $\Phi_\gamma(z) := \int_0^z \gamma(s) \mathrm{d}s$. The function $\Phi_\gamma(z)$ is convex because $\Phi''_\gamma(z) = \gamma'(z) \geqslant 0$. One can check that

$$\gamma(z_1)(z_2 - z_1) \leqslant \Phi_\gamma(z_2) - \Phi_\gamma(z_1) \leqslant \gamma(z_2)(z_2 - z_1), \qquad \forall z_1, z_2 \in \mathrm{R}. \tag{27}$$

According to $\gamma(0) = 0$ and (27), a function $\tilde{\Phi}_\gamma(z)$ can be defined such that

$$\tilde{\Phi}_\gamma(z) := z\gamma(z) - \Phi_\gamma(z) \geqslant 0, \qquad \forall z \in \mathrm{R}.$$

Some estimates on the function u_i are deduced in the following lemma. These a priori estimates will serve as uniform bounds in order to prove convergence.

Lemma 4.1. *Let the assumptions of Lemma 3.7 be fulfilled. Then there exists a positive constant C such that*

(i) $\sum_{i=1}^n \|u_i\|_V^2 \tau \leqslant C$;

(ii) $\max_{1 \leqslant i \leqslant n} \|u_i\|_V^2 + \sum_{i=1}^n \|u_i - u_{i-1}\|_V^2 \leqslant C$ *if* $\boldsymbol{a}(u) = \boldsymbol{a}$

and $\sum_{i=1}^n \|\delta u_i\|^2 \tau + \max_{1 \leqslant i \leqslant n} \|u_i\|_V^2 + \sum_{i=1}^n \|u_i - u_{i-1}\|_V^2 \leqslant C$ *if* $\boldsymbol{a}(u) \neq \boldsymbol{a}$;

(iii) $\max_{1 \leqslant i \leqslant n} \|\theta(u_i)\|_V^2 \leqslant C$;

(iv) $\max_{1 \leqslant i \leqslant n} \|\delta\theta(u_i)\|_{V^*} \leqslant C$ *and* $\sum_{i=1}^n \|\delta\theta(u_i)\|_{V^*}^2 \tau \leqslant C$;

Proof. (i) Setting $\varphi = u_i$ into (26), multiplying by τ and summing it up for $i = 1, \ldots, j$

we have

$$\sum_{i=1}^{j} (\delta\theta(u_i), u_i)\, \tau + \sum_{i=1}^{j} (K_i \nabla u_i, \nabla u_i)\, \tau = \sum_{i=1}^{j} (f_i, u_i)\, \tau - \sum_{i=1}^{j} h_i u_i|_{\Gamma_0} \tau - \sum_{i=1}^{j} (\boldsymbol{a}(u_i), \nabla u_i)\, \tau.$$

According to Abel's summation rule, the assumption that θ is monotonically nondecreasing, the growth condition on θ and $u_0 \in L_2(\Omega)$, we can derive a lower bound for the first term on the left-hand side (LHS) of the above equation:

$$\begin{aligned}
\sum_{i=1}^{j} (\theta(u_i) - \theta(u_{i-1}), u_i) &= (\theta(u_j), u_j) - (\theta(u_0), u_0) - \sum_{i=1}^{j} (\theta(u_{i-1}), u_i - u_{i-1}) \\
&\overset{(27)}{\geqslant} (\theta(u_j), u_j) - (\theta(u_0), u_0) - \sum_{i=1}^{j} \int_\Omega [\Phi_\theta(u_i) - \Phi_\theta(u_{i-1})] \\
&= \left[(\theta(u_j), u_j) - \int_\Omega \Phi_\theta(u_j) \right] - \left[(\theta(u_0), u_0) - \int_\Omega \Phi_\theta(u_0) \right] \\
&= \int_\Omega \tilde{\Phi}_\theta(u_j) - \int_\Omega \tilde{\Phi}_\theta(u_0) \\
&\geqslant - \int_\Omega u_0 \theta(u_0) \\
&\geqslant -C.
\end{aligned}$$

On the first term of the right-hand side, we apply the Cauchy and Young inequalities and the Friedrichs inequality to obtain

$$\left| \sum_{i=1}^{j} (f_i, u_i)\, \tau \right| \leqslant C_\varepsilon \sum_{i=1}^{j} \|f_i\|^2 \tau + \varepsilon \sum_{i=1}^{j} \|u_i\|^2 \tau \leqslant C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau.$$

In the same way, using the trace theorem, one can prove that

$$\left| \sum_{i=1}^{j} h_i u_i|_{\Gamma_0} \tau \right| \leqslant C_\varepsilon \sum_{i=1}^{j} |h_i|^2 \tau + \varepsilon \sum_{i=1}^{j} \|u_i\|^2_{L_2(\Gamma_0)} \tau \leqslant C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau;$$

$$\left| \sum_{i=1}^{j} (\boldsymbol{a}(u_i), \nabla u_i)\, \tau \right| \leqslant C_\varepsilon \sum_{i=1}^{j} \|\boldsymbol{a}(u_i)\|^2 \tau + \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau \leqslant C_\varepsilon + \varepsilon \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau.$$

After fixing a sufficiently small positive ε, an application of the Friedrichs inequality concludes the proof.

(ii) Choosing $\varphi = \delta u_i$ into (26), multiplying by τ and summing up over $i = 1, \ldots, j$ one can get

$$\sum_{i=1}^{j} (\delta\theta(u_i), \delta u_i)\, \tau + \sum_{i=1}^{j} (K_i \nabla u_i, \nabla \delta u_i)\, \tau = \sum_{i=1}^{j} (f_i, \delta u_i)\, \tau - \sum_{i=1}^{j} h_i \delta u_i|_{\Gamma_0} \tau - \sum_{i=1}^{j} (\boldsymbol{a}(u_i), \nabla \delta u_i)\, \tau.$$

Using the monotonicity of θ and the mean value theorem, the first term on the LHS can be estimated by

$$\sum_{i=1}^{j} (\delta\theta(u_i), \delta u_i)\,\tau = \sum_{i=1}^{j} \frac{1}{\tau}\,(\theta(u_i) - \theta(u_{i-1}), u_i - u_{i-1}) \geqslant 0 \qquad \text{if } \boldsymbol{a}(u) = \boldsymbol{a};$$

$$\sum_{i=1}^{j} (\delta\theta(u_i), \delta u_i)\,\tau \geqslant \frac{\theta_0}{\tau} \sum_{i=1}^{j} \|u_i - u_{i-1}\|^2 = \theta_0 \sum_{i=1}^{j} \|\delta u_i\|^2\,\tau \qquad \text{if } \boldsymbol{a}(u) \neq \boldsymbol{a}.$$

The second term on the LHS can be rewritten using Abel's summation rule, namely

$$\sum_{i=1}^{j} (K_i \nabla u_i, \nabla \delta u_i)\,\tau \geqslant \frac{C_0 D_0}{2} \left(\|\nabla u_j\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^{j} \|\nabla (u_i - u_{i-1})\|^2 \right).$$

We apply Abel's summation rule, the Cauchy and Young inequalities, the trace theorem, (13), the Friedrichs inequality (3) and Lemma 4.1(i) on the second term of the RHS to get

$$\begin{aligned}
\left| \sum_{i=1}^{j} h_i \delta u_i|_{\Gamma_0} \tau \right| &= \left| u_j|_{\Gamma_0} h_j - u_0|_{\Gamma_0} h_0 - \sum_{i=1}^{j} \delta h_i u_{i-1}|_{\Gamma_0} \tau \right| \\
&\leqslant \varepsilon \|u_j\|^2_{L_2(\Gamma_0)} + C_\varepsilon |h_j|^2 + C + C \sum_{i=1}^{j} |\delta h_i|^2 \tau + C \sum_{i=1}^{j} \|u_{i-1}\|^2_{L_2(\Gamma_0)}\,\tau \\
&\leqslant C_\varepsilon + \varepsilon \|\nabla u_j\|^2 + C \|\nabla u_0\|^2 \tau + C \sum_{i=1}^{j} \|\nabla u_i\|^2 \tau \\
&\leqslant C_\varepsilon + \varepsilon \|\nabla u_j\|^2,
\end{aligned}$$

when $\tau < \tau_0$. Analogue one can prove that for $\tau < \tau_0$, it holds that

$$\sum_{i=1}^{j} (f_i, \delta u_i)\,\tau = (f_j, u_j) - (f_0, u_0) - \sum_{i=1}^{j} (\delta f_i, u_{i-1})\,\tau \leqslant C_\varepsilon + \varepsilon \|\nabla u_j\|^2.$$

If $\boldsymbol{a}(u) = \boldsymbol{a}$, collecting all considerations above results in

$$\frac{C_0 D_0}{2} \left(\|\nabla u_j\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^{j} \|\nabla (u_i - u_{i-1})\|^2 \right) \leqslant C_\varepsilon + \varepsilon \|\nabla u_j\|^2.$$

Secondly, we discuss the case $\boldsymbol{a}(u) \neq \boldsymbol{a}$. Also the following partial summation formula is satisfied

$$\sum_{i=1}^{j} (\boldsymbol{a}(u_i), \nabla \delta u_i)\,\tau = (\boldsymbol{a}(u_j), \nabla u_j) - (\boldsymbol{a}(u_0), \nabla u_0) - \sum_{i=1}^{j} (\delta \boldsymbol{a}(u_i), \nabla u_{i-1})\,\tau.$$

Employing the mean value theorem, assumption (8) and Lemma 4.1(i), we get for $\tau < \tau_0$

$$\left| \sum_{i=1}^{j} (\boldsymbol{a}(u_i), \nabla \delta u_i)\,\tau \right| \leqslant C_\varepsilon + \varepsilon \|\nabla u_j\|^2 + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau.$$

We arrive at

$$\theta_0 \sum_{i=1}^{j} \|\delta u_i\|^2 \tau + \frac{C_0 D_0}{2} \left(\|\nabla u_j\|^2 + \sum_{i=1}^{j} \|\nabla (u_i - u_{i-1})\|^2 \right) \leqslant C_\varepsilon + \varepsilon \|\nabla u_j\|^2 + \varepsilon \sum_{i=1}^{j} \|\delta u_i\|^2 \tau.$$

Fixing a sufficiently small ε and applying the Friedrichs inequality in both cases give the estimates.

(iii) This follows immediately from Lemma 4.1(ii) coupled with the growth condition on θ.

(iv) The relation (26) can be rewritten for $\varphi \in V$ as

$$(\delta\theta(u_i), \varphi) = (f_i, \varphi) - (K_i \nabla u_i, \nabla\varphi) - (\boldsymbol{a}(u_i), \nabla\varphi) - h_i \varphi|_{\Gamma_0}.$$

A standard argumentation yields

$$|\, (\delta\theta(u_i), \varphi) \,| \lesssim (1 + \|f_i\| + |h_i| + \|\nabla u_i\|) \|\varphi\|_V ,$$

which implies

$$\|\delta\theta(u_i)\|_{V^*} = \sup_{\substack{\varphi \in V \\ \|\varphi\|_V \leqslant 1}} |\, (\delta\theta(u_i), \varphi) \,| \lesssim 1 + \|f_i\| + |h_i| + \|\nabla u_i\| . \tag{28}$$

An application of Lemma 4.1(ii) gives the first inequality. Taking the second power in (28), multiplying the inequality by τ, summing it up for $i = 1, \ldots, j$, and applying Lemma 4.1(i), we get the second inequality. □

The existence of a weak solution will be proved using Rothe's method. The variational formulation of (1)-(2) reads as: find (K, u) such that

$$(\partial_t \theta(u), \varphi) + (K \nabla u + \boldsymbol{a}(u), \nabla\varphi) + h\varphi|_{\Gamma_0} = (f, \varphi) \qquad \varphi \in V \tag{29a}$$

$$u|_{\Gamma_0} = U. \tag{29b}$$

Now, let us introduce the following piecewise linear in time functions θ_n and u_n

$$\begin{array}{ll} \theta_n(0) = \theta(u_0) & \\ \theta_n(t) = \theta(u_{i-1}) + (t - t_{i-1})\delta\theta(u_i) & \text{for } t \in (t_{i-1}, t_i]; \\ u_n(0) = u_0 & \\ u_n(t) = u_{i-1} + (t - t_{i-1})\delta u_i & \text{for } t \in (t_{i-1}, t_i]; \end{array}$$

and the step functions $\overline{u}_n$ and $\overline{\theta}_n$

$$\begin{array}{lll} \overline{u}_n(0) = u_0, & \overline{u}_n(t) = u_i, & \text{for } t \in (t_{i-1}, t_i]; \\ \overline{\theta}_n(0) = \theta(u_0), & \overline{\theta}_n(t) = \theta(u_i), & \text{for } t \in (t_{i-1}, t_i]. \end{array}$$

Similarly, the functions $\overline{K}_n, \overline{h}_n, \overline{U}_n$ and $\overline{f}_n$ are defined. The variational formulation (26) can be rewritten as ($t \in (0, T)$)

$$(\partial_t \theta_n(t), \varphi) + \left(\overline{K}_n(t) \nabla \overline{u}_n(t) + \boldsymbol{a}(\overline{u}_n(t)), \nabla\varphi \right) + \overline{h}_n(t)\varphi|_{\Gamma_0} = \left(\overline{f}_n(t), \varphi \right) \qquad \varphi \in V \tag{30a}$$

$$\overline{u}_n|_{\Gamma_0} = \overline{U}_n. \tag{30b}$$

Now, (29) follows after passage to the limit as $\tau \to 0$ in (30). In the case that $\boldsymbol{a}$ is independent of u, the assumptions made so far are not strong enough to prove convergence. Namely, to establish convergence, the strong convergence of $\overline{u}_n$ in $L_2((0,T), L_2(\Omega))$ is needed. The preceding observation, leads to the assumption

$$\theta'(s) \geqslant \theta_0 > 0 \qquad \text{a.e. in R.}$$

Corollary 4.2. *(i) There exists a $w \in C([0,T], V^*) \cap L_\infty((0,T), L_2(\Omega))$ with $\partial_t w \in L_2((0,T), V^*)$ (i.e. w is a.e. differentiable in $(0,T)$). Moreover, there exists a subsequence of θ_n (denoted by θ_n again) such that ($t \in (0,T)$)*

$$\begin{array}{llll} \theta_n \to w & \text{in } C([0,T], V^*), & \overline{\theta}_n(t) \rightharpoonup w(t) & \text{in } L_2(\Omega), \\ \theta_n(t) \rightharpoonup w(t) & \text{in } L_2(\Omega), & \partial_t \theta_n \rightharpoonup \partial_t w & \text{in } L_2((0,T), V^*). \end{array}$$

(ii) Let $\theta' \geqslant \theta_0 > 0$. There exists a $u \in C([0,T], L_2(\Omega)) \cap L_\infty((0,T), L_2(\Omega))$ with $\partial_t u \in L_2((0,T), L_2(\Omega))$ (i.e. u is a.e. differentiable in $(0,T)$). Moreover, there exists a subsequence of u_n (denoted by u_n again) such that ($t \in (0,T)$)

$$\begin{array}{llll} u_n \to u & \text{in } C([0,T], L_2(\Omega)), & \overline{u}_n(t) \rightharpoonup u(t) & \text{in } V, \\ u_n(t) \rightharpoonup u(t) & \text{in } V, & \partial_t u_n \rightharpoonup \partial_t u & \text{in } L_2((0,T), L_2(\Omega)). \end{array}$$

Proof. Thanks to the Rellich-Kondrachov Compactness Theorem [16, Theorem 1, p. 272], we have that

$$V \hookrightarrow\hookrightarrow L_2(\Omega) \cong (L_2(\Omega))^* \hookrightarrow\hookrightarrow V^*.$$

(i) Applying Lemma 4.1(*iii*) and (*iv*), we get

$$\max_{t\in[0,T]} \left\| \overline{\theta}_n(t) \right\|^2 + \int_0^T \|\partial_t \theta_n\|_{V^*}^2 \leqslant C.$$

Consequently, an application of [13, Lemma 1.3.13] gives the proof.

(ii) Applying Lemma 4.1(*ii*), we have

$$\max_{t\in[0,T]} \|\overline{u}_n(t)\|_V^2 + \int_0^T \|\partial_t u_n\|^2 \leqslant C.$$

Hence, an application of [13, Lemma 1.3.13] concludes the proof. □

Finally, the theorem to be proved is the following.

Theorem 4.3. *Let the assumptions of Lemma 3.7 be fulfilled. Suppose that there exists a positive real constant θ_0 such that $\theta' \geqslant \theta_0 > 0$. Then there exists a weak solution to (29).*

Proof. Take any $\xi \in (0,T)$ and integrate (30*a*) on $(0,\xi)$ to get for each $\varphi \in V$

$$\int_0^\xi (\partial_t \theta_n(t), \varphi) + \int_0^\xi \left(\overline{K}_n(t) \nabla \overline{u}_n(t) + \boldsymbol{a}(\overline{u}_n(t)), \nabla\varphi \right) + \int_0^\xi \overline{h}_n(t) \varphi|_{\Gamma_0} = \int_0^\xi \left(\overline{f}_n(t), \varphi \right). \quad (31)$$

We have to pass to the limit for $n \to \infty$ in (31). Each term in (31) is considered separately. The Rothe sequence $\{\overline{u}_n\}_{n\in\mathbb{N}}$ is bounded in the space $L_2((0,T),V)$ thanks to Lemma 4.1(i), indeed

$$\|\overline{u}_n\|^2_{L_2((0,T),V)} = \int_0^T \|\overline{u}_n(t)\|^2_V \,\mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\overline{u}_n(t)\|^2_V \,\mathrm{d}t = \sum_{i=1}^n \|u_i\|^2_V \tau \leqslant C.$$

Therefore, the reflexivity of $L_2((0,T),V)$ implies (for a subsequence denoted by $\overline{u}_n$ again) the following useful result

$$\overline{u}_n \rightharpoonup u \qquad \text{in } L_2((0,T),V). \tag{32}$$

Thus $u \in L_2((0,T),V) \cap C([0,T], L_2(\Omega))$. Firstly, we apply the Minty-Browder's trick [16] to prove that $w = \theta(u)$. It holds that for $t \in (t_{i-1}, t_i]$

$$\left(\theta_n - \overline{\theta}_n\right)(t) = \theta(u_{i-1}) + \left(\frac{t - t_{i-1}}{\tau}\right)(\theta(u_i) - \theta(u_{i-1})) - \theta(u_i) = (t - t_i)\partial_t\theta_n(t).$$

Hence, using Lemma 4.1(iv), we obtain for $t \in (t_{i-1}, t_i]$

$$\left\|\left(\theta_n - \overline{\theta}_n\right)(t)\right\|_{V^*} \leqslant \tau \left\|\partial_t\theta_n(t)\right\|_{V^*} \lesssim \tau.$$

Therefore, according to Corollary 4.2(i)

$$\overline{\theta}_n \to w \qquad \text{in } C([0,T], V^*). \tag{33}$$

Thanks to the monotonicity of the function θ, it yields for any fixed $v \in L_2((0,T), L_2(\Omega))$ that

$$\int_0^T (\theta(\overline{u}_n) - \theta(v), \overline{u}_n - v) \geqslant 0.$$

Due to (32) and (33), we have that, for $\tau \to 0$,

$$\int_0^T (w - \theta(v), u - v) \geqslant 0.$$

Firstly, suppose that $v = u + \varepsilon z$ with $\varepsilon > 0$ and $z \in L_2((0,T), L_2(\Omega))$. We get

$$\int_0^T (w - \theta(u + \varepsilon z), -\varepsilon z) \geqslant 0.$$

Next, dividing this equation by $-\varepsilon$, taking the limit $\varepsilon \to 0$ and using the continuity of θ, we arrive at

$$\int_0^T (w - \theta(u), z) \leqslant 0.$$

Secondly, assume that $v = u - \varepsilon z$ with $\varepsilon > 0$ and $z \in L_2((0,T), L_2(\Omega))$. Then

$$\int_0^T (w - \theta(u), z) \geqslant 0.$$

Combining the previous results give us

$$\int_0^T (w - \theta(u), z) = 0, \qquad \forall z \in L_2((0,T), L_2(\Omega)).$$

Therefore, $w = \theta(u) \in L_2((0,T), L_2(\Omega))$. Applying Corollary 4.2(i) gives for all $\varphi \in V$ that

$$\left| \int_0^\xi (\partial_t \theta_n(t), \varphi) - \int_0^\xi (\partial_t \theta(u(t)), \varphi) \right| \to 0 \qquad \text{if } n \to \infty.$$

Now, we focus on the second term in the LHS. This term can be split into two parts. Firstly, we consider the second part. The sequences u_n and $\overline{u}_n$ have the same limit in the space $L_2((0,T), V)$. Employing Lemma 4.1(ii) gives

$$\begin{aligned}
\lim_{n\to\infty} \|u_n - \overline{u}_n\|^2_{L_2((0,T),V)} &= \lim_{n\to\infty} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|(u_n - \overline{u}_n)(t)\|_V^2 \, \mathrm{d}t \\
&= \lim_{n\to\infty} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left\| u_{i-1} + \frac{t - t_{i-1}}{\tau}(u_i - u_{i-1}) - u_i \right\|_V^2 \mathrm{d}t \\
&\leqslant 4 \lim_{n\to\infty} \sum_{i=1}^n \|u_i - u_{i-1}\|_V^2 \, \tau \\
&\leqslant \lim_{n\to\infty} \frac{C}{n} = 0. \qquad (34)
\end{aligned}$$

Therefore, if $\boldsymbol{a}(u) \neq \boldsymbol{a}$, it holds, thanks to assumption (8) and Corollary 4.2(ii), that for all $\varphi \in V$

$$\left| \int_0^\xi (\boldsymbol{a}(\overline{u}_n(t)) - \boldsymbol{a}(u(t)), \nabla\varphi) \right| \lesssim \|\nabla \varphi\| \sqrt{T} \sqrt{\int_0^T \|\overline{u}_n(t) - u(t)\|^2} \to 0 \text{ if } n \to \infty.$$

Secondly, applying the Green theorem and taking $\varphi \in \{\psi \in C^\infty(\Omega) : \psi|_{\Gamma_D} = 0, \psi|_{\Gamma_0} = \mathit{constant}\}$ in the first part, we deduce that

$$\begin{aligned}
&\int_0^\xi \left(\overline{K}_n \nabla \overline{u}_n, \nabla \varphi \right) \\
&= \int_0^\xi \overline{k}_n \left(\overline{\kappa}_n \nabla \overline{u}_n, \nabla\varphi \right)_{\Omega_0} + \int_0^\xi \left(\overline{\kappa}_n \nabla \overline{u}_n, \nabla\varphi \right)_{\Omega\setminus\Omega_0} \\
&= \int_0^\xi \overline{k}_n \left(\overline{u}_n, \overline{\kappa}_n \nabla\varphi \cdot \boldsymbol{\nu} \right)_{\partial\Omega_0} - \int_0^\xi \overline{k}_n \left(\overline{u}_n, \nabla \cdot (\overline{\kappa}_n \nabla \varphi) \right)_{\Omega_0} + \int_0^\xi \left(\overline{\kappa}_n \nabla \overline{u}_n, \nabla\varphi \right)_{\Omega\setminus\Omega_0} \\
&= \int_0^\xi \overline{k}_n \left(\overline{u}_n, \overline{\kappa}_n \nabla\varphi \cdot \boldsymbol{\nu} \right)_{\partial\Omega_0} - \int_0^\xi \overline{k}_n \left(\overline{u}_n, \nabla \overline{\kappa}_n \cdot \nabla\varphi + \overline{\kappa}_n \Delta \varphi \right)_{\Omega_0} + \int_0^\xi \left(\overline{\kappa}_n \nabla \overline{u}_n, \nabla\varphi \right)_{\Omega\setminus\Omega_0}.
\end{aligned} \qquad (35)$$

At this point, we need some auxiliary results. In light of equation (32) and Lemma 4.1(i), applying the Nečas inequality [17]

$$\|z\|_\Gamma^2 \leqslant \varepsilon \|\nabla z\|^2 + C_\varepsilon \|z\|^2, \qquad \forall z \in H^1(\Omega), \qquad 0 < \varepsilon < \varepsilon_0,$$

implies

$$\int_0^T \|\overline{u}_n - u\|_\Gamma^2 \leqslant \varepsilon \int_0^T \|\nabla(\overline{u}_n - u)\|^2 + C_\varepsilon \int_0^T \|\overline{u}_n - u\|^2 \leqslant \varepsilon + C_\varepsilon \int_0^T \|\overline{u}_n - u\|^2 .$$

Passing to the limit for $\tau \to 0$ and applying Corollary 4.2(*ii*) we obtain

$$\lim_{\tau\to 0} \int_0^T \|\overline{u}_n - u\|_\Gamma^2 \leqslant \varepsilon,$$

which is valid for any small $\varepsilon > 0$. Hence

$$\lim_{\tau\to 0} \int_0^T \|\overline{u}_n - u\|_\Gamma^2 = 0 \qquad \text{and} \qquad \overline{u}_n \to u \text{ a.e. in } (0,T) \times \Gamma.$$

Repeating this consideration for Ω_0 instead of Ω gives

$$\lim_{\tau\to 0} \int_0^T \|\overline{u}_n - u\|_{\partial\Omega_0}^2 = 0 \qquad \text{and} \qquad \overline{u}_n \to u \text{ a.e. in } (0,T) \times \partial\Omega_0. \tag{36}$$

Due to the construction we have that $C_0 \leqslant \overline{k}_n \leqslant C_1$. This yields that $\overline{k}_n \rightharpoonup k$ (for a subsequence) in $L_2((0,T))$. Therefore, employing $\overline{k}_n \rightharpoonup k$, (10), (32), (34) and (36), we get after passage to the limit as $\tau \to 0$ in (35) that

$$\begin{aligned}
&\lim_{\tau\to 0} \int_0^\xi \left(\overline{K}_n \nabla \overline{u}_n, \nabla\varphi\right) \\
&= \int_0^\xi k\,(u, \kappa\nabla\varphi\cdot\boldsymbol{\nu})_{\partial\Omega_0} - \int_0^\xi k\,(u, \nabla\kappa\cdot\nabla\varphi + \kappa\Delta\varphi)_{\Omega_0} + \int_0^\xi (\kappa\nabla u, \nabla\varphi)_{\Omega\setminus\Omega_0} \\
&= \int_0^\xi k\,(u, \kappa\nabla\varphi\cdot\boldsymbol{\nu})_{\partial\Omega_0} - \int_0^\xi k\,(u, \nabla\cdot(\kappa\nabla\varphi))_{\Omega_0} + \int_0^\xi (\kappa\nabla u, \nabla\varphi)_{\Omega\setminus\Omega_0} \\
&= \int_0^\xi k\,(\kappa\nabla u, \nabla\varphi)_{\Omega_0} + \int_0^\xi (\kappa\nabla u, \nabla\varphi)_{\Omega\setminus\Omega_0} \\
&= \int_0^\xi (K\nabla u, \nabla\varphi)\,.
\end{aligned}$$

Applying the density argument $\overline{\{\psi \in C^\infty(\Omega) : \psi|_{\Gamma_D} = 0,\ \psi|_{\Gamma_0} = \mathit{constant}\}} = V$, we conclude that

$$\lim_{\tau\to 0} \int_0^\xi \left(\overline{K}_n \nabla \overline{u}_n, \nabla\varphi\right) = \int_0^\xi (K\nabla u, \nabla\varphi)\,, \qquad \forall\varphi \in V.$$

Moreover, due the continuity of h and f, one can deduce that, for any $\varphi \in V$,

$$\left| \int_0^\xi \left(\overline{h}_n(t) - h(t)\right) \varphi|_{\Gamma_0} \right| \lesssim \|\varphi\|_V \int_0^T \left|\overline{h}_n(t) - h(t)\right| \lesssim \tau$$

$$\left| \int_0^\xi \left(\overline{f}_n(t) - f(t), \varphi\right) \right| \leqslant \|\varphi\| \int_0^T \left\|\overline{f}_n(t) - f(t)\right\| \lesssim \tau.$$

Collecting all considerations above and passing to the limit for $\tau \to 0$ in (31) we arrive at

$$\int_0^\xi (\partial_t\theta(u), \varphi) + \int_0^\xi (K\nabla u + \boldsymbol{a}(u), \nabla\varphi) + \int_0^\xi h\varphi|_{\Gamma_0} = \int_0^\xi (f, \varphi) \qquad \varphi \in V.$$

This is valid for any $\xi \in (0,T)$. Differentiation with respect to ξ gives (29a). Taking the limit $\tau \to 0$ in (30b) and using the continuity of U we get (29b), which concludes the proof. □

4.2 *Second solution method*

Consider the notations in Subsection 3.2. The variational formulation of (19) on timestep t_i for all $\varphi \in W$ is

$$\begin{aligned}(\delta\vartheta(v_i), \varphi) + \left(K_i \nabla (v_i + \tilde{U}_i) + \boldsymbol{b}(v_i), \nabla\varphi\right) &= (f_i, \varphi)\,, \\ \int_{\Gamma_0} \left(-K_i \nabla (v_i + \tilde{U}_i) - \boldsymbol{b}(v_i)\right) \cdot \boldsymbol{\nu} &= h_i,\end{aligned} \tag{37}$$

with $u_i = v_i + \tilde{U}_i$. Using analogous notations as in the previous Subsection 4.1, the variational formulation (37) can be rewritten as

$$(\partial_t \vartheta_n, \varphi) + \left(\overline{K}_n \nabla (\overline{v}_n + \overline{\tilde{U}}_n) + \boldsymbol{a}(\overline{v}_n), \nabla\varphi\right) = \left(\overline{f}_n, \varphi\right), \tag{38a}$$

$$\int_{\Gamma_0} \left(-\overline{K}_n \nabla (\overline{v}_n + \overline{\tilde{U}}_n) - \boldsymbol{b}(\overline{v}_n)\right) \cdot \boldsymbol{\nu} = \overline{h}_n, \tag{38b}$$

where $\varphi \in W$ and $\overline{u}_n = \overline{v}_n + \overline{\tilde{U}}_n$. The variational formulation of (1)-(2) for all $\varphi \in W$ reads as: find (K, u) with $u = v + \tilde{U}$ such that

$$(\partial_t \vartheta(v), \varphi) + \left(K \nabla (v + \tilde{U}) + \boldsymbol{b}(v), \nabla\varphi\right) = (f, \varphi)\,, \tag{39a}$$

$$\int_{\Gamma_0} \left(-K \nabla (v + \tilde{U}) - \boldsymbol{b}(v)\right) \cdot \boldsymbol{\nu} = h. \tag{39b}$$

Analysis similar to that in the previous Subsection 4.1 shows that the limit $\tau \to 0$ in (38a) results into (39a). There is an extra term to take under consideration, namely ($\xi \in (0,T)$ and $\varphi \in W$)

$$\left| \int_0^\xi \left(\overline{k}_n \overline{\kappa}_n \nabla \overline{\tilde{U}}_n - k\kappa \nabla \tilde{U}, \nabla\varphi\right) \right| \to 0.$$

The convergence of (38b) to (39b) for any $t \in (0,T)$ follows from the continuity of h.

This subsection concludes with an analogue of Theorem 4.3.

Theorem 4.4. *Let the assumptions of Lemma* 3.7 *be fulfilled. Moreover, suppose that there exist real constants* θ_0 *and* θ_1 *such that* $0 < \theta_0 < \theta' \leqslant \theta_1$. *Then there exists a weak solution to (39).*

5. Numerical experiments

The aim of the simulations is to analyze both algorithms proposed in Sections 3 and 4. The 2D Finite Elements code Freefem++ is used.

The domain under consideration is the rectangle $\Omega = \left(-\frac{1}{2}, 1\right) \times (-1,1)$, with $\Omega_0 = \left(-\frac{1}{2}, 0\right) \times (-1,1)$ in R^2. The time interval is $[0,1]$, i.e., $T = 1$. The boundary Γ is split into three nonoverlapping parts, namely Γ_D (right), Γ_N (top and bottom) and Γ_0 (left part of Γ), see Fig. 1.

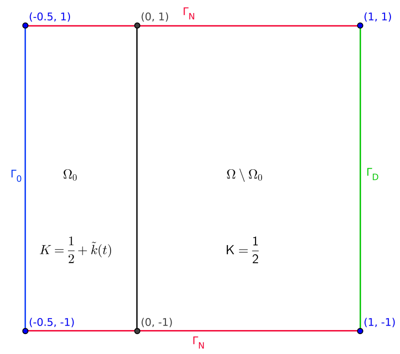


Figure 1. The domain used in the numerical experiments.

In the experiments, the exact solution (K, u) is prescribed as follows

$$K(t, x, y) = (1 + \sin(10t))\, \mathbb{1}_{\{x<0\}} + \frac{1}{2}; \qquad u(t, x, y) = (1 + t) \sin\left(\frac{\pi}{2}(1 - x)\right).$$

The function $\mathbb{1}_{\{x<0\}}$ is the indicator function of the subset $\left\{(x, y) \in \mathrm{R}^2 : x < 0\right\}$ of R^2. Moreover, it is assumed that

$$\theta(s) = \sqrt{s^2 + 1} \qquad \text{and} \qquad \boldsymbol{a} \equiv \mathbf{0}.$$

Some simple calculations give the exact data for the numerical experiment

$$g^D = g^N = 0; \qquad U(t) = \frac{1 + t}{\sqrt{2}}; \qquad h(t) = \frac{\pi}{\sqrt{2}} (1 + t) (1.5 + \sin(10t)).$$

This section is split into two subsections. The first two are devoted to a different solution method. The results are summarized in the third subsection. The purpose is the recovery of

$$\tilde{k}(t) := 1 + \sin(10t).$$

For the time discretization an equidistant time partitioning is chosen with time step $\tau = 0.02$ and for the space discretization a fixed uniform mesh is used consisting of 144528 triangles.

5.1 *First solution method*

An uncorrelated noise is added to the additional condition $U(t)$ in order to simulate the errors present in real measurements. The noise is generated randomly with given magnitude $e = 0\%$, 0.5%, and 1%. Applying the backward Euler difference scheme into (20), a recurrent system of nonlinear elliptic BVPs for $(K_i, u_i) \approx (K(t_i), u(t_i))$, $i = 1, 2, \ldots, 50$ and $\varphi \in V$ have to be solved

$$\frac{1}{\tau} (\theta(u_i), \varphi) + (K_i \nabla u_i, \nabla\varphi) = (f_i, \varphi) - h_i \varphi|_{\Gamma_0} + \frac{1}{\tau} (\theta(u_{i-1}), \varphi); \qquad u_0 = u_0; \qquad (40)$$

with

$$(f_i, \varphi) = \left(\frac{(1+t_i)\left(\sin\left(\frac{\pi}{2}(1-x)\right)\right)^2}{\sqrt{(1+t_i)^2\left(\sin\left(\frac{\pi}{2}(1-x)\right)\right)^2 + 1}}, \varphi \right)$$
$$+ \left((1.5 + \sin(10t_i)) \left(\frac{\pi}{2}\right)^2 (1+t_i)\sin\left(\frac{\pi}{2}(1-x)\right), \varphi \right)_{\Omega_0}$$
$$+ \left(0.5 \left(\frac{\pi}{2}\right)^2 (1+t_i)\sin\left(\frac{\pi}{2}(1-x)\right), \varphi \right)_{\Omega\setminus\Omega_0}. \quad (41)$$

On every time level, Newton's method is applied to deal with the nonlinearity. More precisely, given a solution u_i^l from iteration l, a perturbation du_i is searched such that

$$u_i^{l+1} = u_i^l + du_i$$

fulfills the nonlinear problem (40). The perturbation du_i is zero at the boundaries where the Dirichlet conditions are applied. Inserting u_i^{l+1} in the BVP (40), the θ term in the LHS is linearized as

$$\theta(u_i^{l+1}) \approx \theta(u_i^l) + \theta'(u_i^l)du_i.$$

As initial guess u_i^0, the solution of the linear problem with $\theta(s) = 1$ and $\boldsymbol{a}(s) = \mathbf{0}$ is taken. The iterations are stopped when $\|u_i^{l+1} - u_i^l\|_{L_\infty(\Omega)} = \|du_i\|_{L_\infty(\Omega)} < 10^{-5}$ or when the number of iterations exceed the limit 25. The unknowns $\tilde{k}_i \approx \tilde{k}(t_i), i = 1, \ldots, 50$, are determined by the nonlinear conjugate gradient method. On each time step $t_i, i = 1, \ldots, 50$, the functional

$$J_1(\tilde{k}_i) := \|u_i - U(t_i)\|^2_{L_2(\Gamma_0)}$$

is minimized. The starting point for this algorithm on the first time step is set as $\tilde{k}_1^0 = 1$. The starting point on the following time steps is $\tilde{k}_{i-1}$, the minimizing value of the functional in the previous time step. The algorithm stops after maximum 10 iterations with the prescribed error tolerance 10^{-6}.

5.2 *Second solution method*

Again, an uncorrelated noise with magnitude $e = 0\%, 0.5\%$ and 1% is added to the additional condition $h(t)$. Utilizing the backward Euler difference scheme into (23), a recurrent system of linear elliptic BVPs for $(K_i, u_i) \approx (K(t_i), u(t_i)), i = 1, 2, \ldots, 50$ and $\varphi \in W$ has to be solved

$$\frac{1}{\tau}(\theta(u_i), \varphi) + (K_i \nabla u_i, \nabla\varphi) = (f_i, \varphi) + \frac{1}{\tau}(\theta(u_{i-1}), \varphi), \qquad u_0 = u_0; \quad (42)$$

where (f_i, φ) is as in (41) and $u_i = \tilde{U}_i$ on Γ_0. The nonlinearity is handled in the same way as in the first solution method. Now, on each time step $t_i, i = 1, \ldots, 50$, the functional

$$J_2(\tilde{k}_i) := \left(-\int_{\Gamma_0} (\tilde{k}_i + 0.5)\nabla u_i \cdot \boldsymbol{\nu} - h(t_i) \right)^2$$

is minimized.

5.3 *Results*

At each time step, the resulting elliptic BVP's (40) and (42) are solved numerically by the finite element method (FEM) using second order (P2-FEM) Lagrange polynomials. The results from the recovery of $\tilde{k}(t)$ using both solution methods for the different values of the amplitude e are shown in Fig. 2, 3 and 4. The exact $\tilde{k}(t_i)$ is denoted by a solid line and the approximations $\tilde{k}_i$ by linespoints; $i = 1, \ldots, 50$. The evolution of the absolute $\tilde{k}_i$-error for the different time steps is shown in Fig. 5.

The experiments demonstrate that the approximation becomes less accurate with increasing magnitude e when the number of time discretization intervals and the number of triangles in the space discretization is fixed. In this experiment, the second solution method is more accurate.

6. Conclusion

In this contribution, a nonlinear parabolic problem of second order with an unknown diffusion coefficient in a subregion is considered. In this subregion, the diffusion coefficient is only time dependent. Two different solution methods are considered. In the first solution method, an additional Dirichlet condition is prescribed on a part of the surface of the region with unknown coefficient. In the second solution method, an additional total flux condition is prescribed through the same surface. First, for both solution methods, the existence of a solution on a single time step is proved using coefficient to data mappings. Afterwards, a numerical algorithm based on Rothe's method is established and the convergence of this scheme is shown. No uniqueness of the solution can be assured. The convergence of the numerical algorithm is illustrated by a numerical experiment.

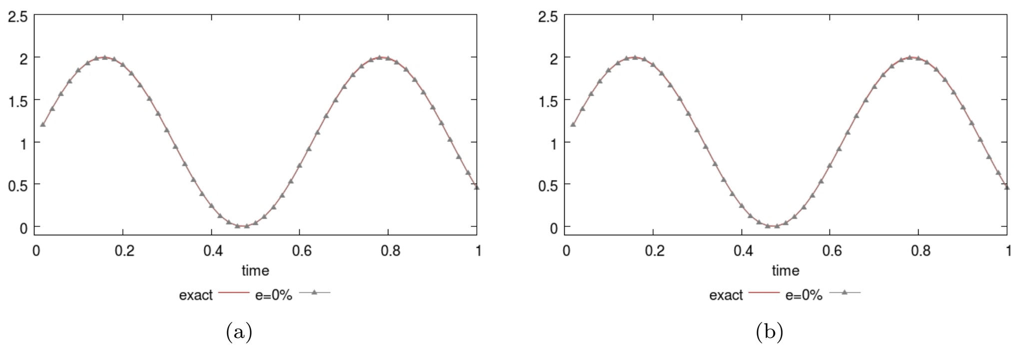
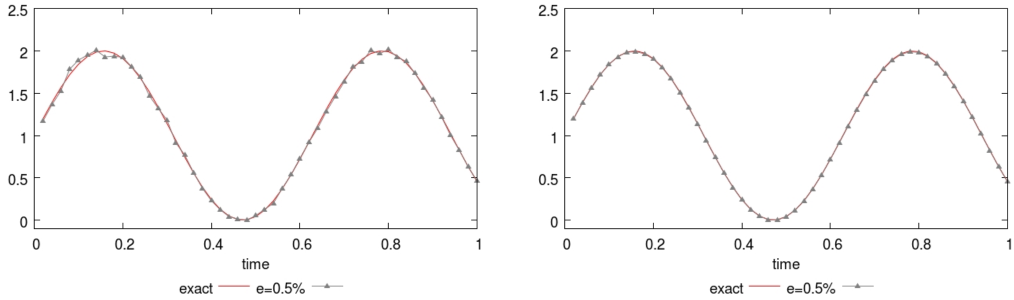
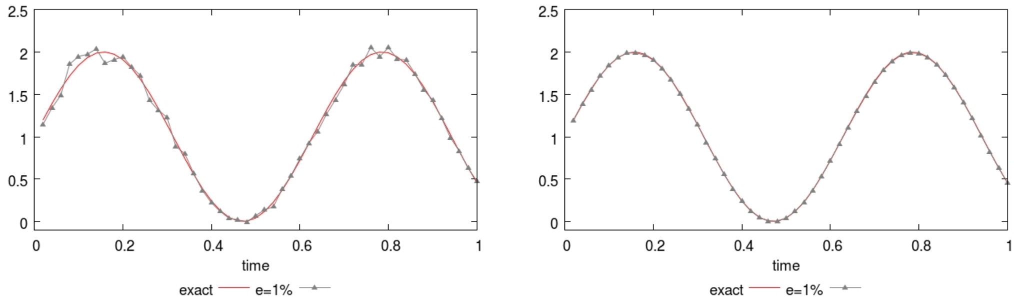


Figure 2. Noise $e = 0\%$: numerical value of $\tilde{k}_i$ using the first solution method (a) and the second solution method (b); $i = 1, \ldots, 50$.



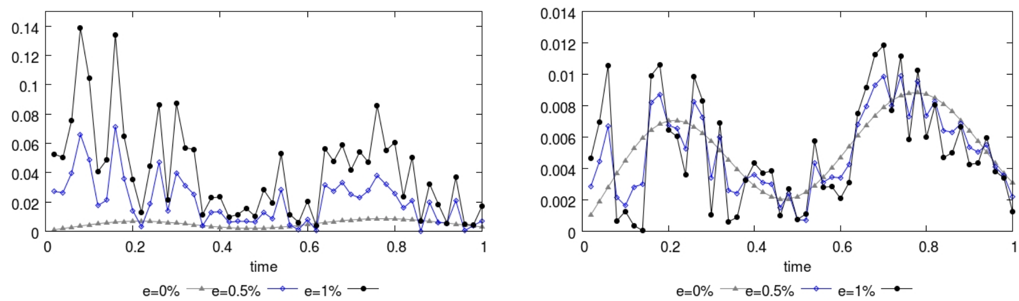
(a) (b)

Figure 3. Noise $e = 0.5\%$: numerical value of $\tilde{k}_i$ using the first solution method (a) and the second solution method (b); $i = 1, \ldots, 50$.



(a) (b)

Figure 4. Noise $e = 1\%$: numerical value of $\tilde{k}_i$ using the first solution method (a) and the second solution method (b); $i = 1, \ldots, 50$.



(a) (b)

Figure 5. The absolute $\tilde{k}_i$-error using the first solution method (a) and using the second solution method (b) for the different noise levels; $i = 1, \ldots, 50$.

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